Finite-size effect in persistence in random walks

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We have investigated the random walk problem in a finite system and studied the crossover induced in the persistence probability by the system size. Analytical and numerical work show that the scaling function is an exponentially decaying function. We consider two cases of trapping, one by a box of size L and the other by a harmonic trap. Our analytic calculations are supported by numerical works. We also present numerical results on the harmonically trapped randomly accelerated particle and the randomly accelerated particle with viscous drag.

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The phenomenon of persistence has attracted a lot of interest in recent years, both theoretically [1–9,24] as well as experimentally [10–13]. The word "persistence" itself conveys the meaning of survival. Associated with this survival is the survival probability $p(t)$. It is simply the probability that the local field has not yet changed its sign upto time t. For a wide range of models the survival probability decays as a power law, that is, $p(t)\sim t^{-\theta}$, where θ is a new nontrivial exponent called the persistence exponent. Established results exist for many models—random walk problem, diffusion problem [1], Ising model with Glauber dynamics [2], surface growth [6], and phase-ordering kinetics [5].

In an experimental setup, however, finite-size effects appear because of the size of the apparatus and boundary effects come into play in the dynamics of the system. As a result the survival probability also depends on the finite-size parameter. A particularly clear example of the crossover effects induced by finite size is the recent experiment on a single polystyrene sphere in a harmonic potential [14]. It is this demonstration of crossover effect that has motivated us in our present work in investigating the finite-size effect on the survival probability and how it scales with the finite-size parameter.

There has been an investigation of this for the Ising system in higher dimension [15], which led to the conclusion that finite-size scaling in the usual sense holds for persistence as well. The system that we investigate here does not show finite-size scaling in exactly that sense, although it exhibits pronounced finite-size effects. We have considered an analytically solvable model in the present work—the case of a Brownian particle confined in a box and the case of a Brownian particle trapped by a harmonic potential. We find that the survival probability $p(t,L)$ does not have the usual scaling form $p(t,L)\sim t^{-\theta}f(\frac{t}{L^z})$, with $f(x)\rightarrow constant$ for $x\rightarrow 0$ and $f(x)\sim x^{\theta}$ for $x\gg 1$. Instead, we find that $p(t,L)$ can be expressed as $p(t,L)=t^{-\theta}f(\frac{t}{L^z})$ but $f(x)\rightarrow 0$ exponentially as $x\rightarrow\infty$. This is consistent with the generic form anticipated by Redner [16]. The exponent z is found to be 2.0 as in the case of Manoj and Ray [15]. It should be noted that this exponential decay for $x\rightarrow\infty$ was actually observed experimentally for step fluctuations [13]. Finite-size study has also been done by Dasgupta *et al.* [17] on step fluctuations and by Constantin *et al.* [18] on nonequilibrium surface growth.

The simplest of all the models for which there exists an established result is the random walk problem. A random walker obeys a differential equation of the form $\frac{dx(t)}{dt}=\eta(t)$, where $\eta(t)$ is a white noise. To find the survival probability we ask the question whether the quantity $\text{sgn}[x(t)-\langle x(t)\rangle]$ has changed its sign upto time t. The survival probability $p(T)$ in terms of the variable $\sigma=\text{sgn}[\bar{X}(T)]$, where $\bar{X}(t)=x(t)/\sqrt{\langle x^2(t)\rangle}$ and $T=\ln(t)$, can be found from $A(T)=\langle\sigma(0)\sigma(T)\rangle$ [1]. In this case $p(T)=(2/\pi)\sin^{-1}[\exp(-\lambda T)]$. For a random walker $\lambda=1/2$ and the survival probability goes as $p(T)\sim\exp(-T/2)$ [19]. Analytical and numerical results show that the probability goes as $p(t)\sim t^{-1/2}$ and the persistence exponent in this case is $\theta=\frac{1}{2}$ [19].

We have investigated the finite-size effect in the random walk problem in two ways. Firstly, the random walker is constrained to move in a box with reflective boundaries at $x=\pm L$. The probability distribution $P(x,t)$ in this case obeys a diffusion equation [20] with an appropriate boundary condition. A solution to the diffusion equation with the proper boundary condition gives $P(x,t)$. In the second problem, the random walker is trapped in a harmonic potential. Both the problems are analogous to each other with the identification $\omega\sim\frac{1}{L}$. In both cases we calculate the correlator $a(t_1,t_2)=\langle x(t_2)x(t_1)\rangle$, where $x(t)$ is the value of x at time t. To make it a Gaussian stationary process (GSP) we transform $x(t)$ to $\bar{X}=x(t)/\sqrt{\langle x^2(t)\rangle}$ and a suitable transformation for the time variable from t to T. Thus the correlator $a(t_2,t_1)\rightarrow f(|T_2-T_1|)$. From the correlator $f(T)$ we get the survival probability $p(t)$.

We first consider a particle in one-dimension performing random walk. The equation governing the dynamics of the particle is given by

$$\frac{dx(t)}{dt}=\eta(t), \tag{1}$$

where $x(t)$ is the displacement of the particle and $\eta(t)$ is a random function. The moments of $\eta(t)$ are given by

$$\langle\eta(t)\rangle=0, \tag{2a}$$

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$$\langle \eta(t)\eta(t')\rangle = D\delta(t-t'), \tag{2b}$$

where D is the diffusion coefficient. In the present problem we confine the motion of the particle within a cage with boundaries at $x=\pm L$. The boundary of the cage is reflective, that is, upon reaching the boundary the particle is reflected to the nearest lattice site.

The probability $P(x,t)$ that the coordinate is x at a time t starting from $x=0$ at $t=0$ obeys the diffusion equation

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{3}$$

with proper boundary condition. Since the particle is reflected from the boundary, the particle current at $x=\pm L$ must be zero. Thus at the boundary we have

$$-D\left.\frac{\partial P(x,t)}{\partial x}\right|_{x=\pm L} = 0. \tag{4}$$

The solution to Eq. (3) with the boundary condition Eq. (4) can be written as

$$P(x,t) = \cos\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2 Dt/L^2}. \tag{5}$$

The complete form of the probability $P(x,t)$ taking into consideration the normalization can be written down as

$$P(x,t) = \frac{1}{2L} + \frac{1}{L}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2 Dt/L^2}, \tag{6}$$

and the average of the square of the displacement is given by

$$\langle(\Delta x)^2\rangle = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n\pi)e^{-n^2\pi^2 Dt/L^2}. \tag{7}$$

This is the exact answer. As expected it exhibits finite size scaling and can be cast in the form

$$\langle(\Delta x)^2\rangle = \frac{L^2}{3}g\left(\frac{t}{L^2}\right), \tag{8}$$

such that $g(\frac{t}{L^2}) \propto \frac{t}{L^2}$ for $\frac{t}{L^2}\to 0$ (i.e., infinite system size) and $g(\frac{t}{L^2})\to 1$ for $\frac{t}{L^2}\to\infty$ that is the extreme case of the finite system.

To see this the sum in Eq. (7) can be decomposed as

$$\langle(\Delta x)^2\rangle = \frac{L^2}{3} - \frac{4L^2}{\pi^2}\left(\sum_{n\ \text{odd}}\frac{1}{n^2}e^{-n^2\pi^2 Dt/L^2} + \sum_{n\ \text{even}}\frac{1}{n^2}e^{-n^2\pi^2 Dt/L^2}\right). \tag{9}$$

In the limit $L\to\infty$, all the modes in the summation of Eq. (7) must be considered. It is then easy to see that

$$\langle(\Delta x)^2\rangle \to \frac{L^2}{3} \quad \text{for } t\to\infty \text{ and } L \text{ finite}, \tag{10}$$

and taking the opposite limit we find that

$$\langle(\Delta x)^2\rangle \to 2Dt \quad \text{for } L\to\infty \text{ and } t \text{ finite}, \tag{11}$$

as expected. Keeping in mind our future need where $f(x)\to 1$ as $x\to 0$ but decays very fast for $x \gg L$, we will express Eq. (7) as an approximate that is easy to handle. This is done using the Euler-Maclaurin sum formula for the two sums in Eq. (9).

Keeping only $e^{-\pi^2 Dt/L^2}$ from among the different exponential decays and working to $O(\frac{t}{L^2})$ the coefficient of $e^{-\pi^2 Dt/L^2}$ the crossover function can be written as

$$\langle(\Delta x)^2\rangle = \frac{L^2}{3} - \frac{2L^2}{\pi^2}\int_1^2 dn\frac{1}{n^2}e^{-n^2\pi^2 Dt/L^2} + \left(\frac{1}{\pi^2}-\frac{1}{3}\right)\left(1+\frac{D\pi^2 t}{L^2}\right)e^{-\pi^2 Dt/L^2}. \tag{12}$$

A more drastic approximation yields the expression

$$\langle(\Delta x)^2\rangle = \frac{L^2}{3}\left[1-\left(1+(\pi^2-6)\frac{Dt}{L^2}\right)e^{-\pi^2 Dt/L^2}\right]. \tag{13}$$

The part in the bracket is the approximation for the function $g(\frac{t}{L^2})$. For $\frac{t}{L^2}\to\infty$, Eq. (13) correctly reduces to $\frac{L^2}{3}$, while for $\frac{t}{L^2}\ll 1$, we expand Eq. (13) to obtain

$$\langle(\Delta x)^2\rangle = 2Dt\left[1-\gamma\frac{Dt}{L^2}\right], \tag{14}$$

where

$$\gamma = \pi^2\left(1-\frac{\pi^2}{12}\right). \tag{15}$$

The leading term in Eq. (14) is the correct limit for the unbounded system and the second term is the first correction for finite L.

We now proceed to calculate the correlator $\langle x(t_2)x(t_1)\rangle$ for the dynamics of Eq. (1) keeping in mind the approximations used in arriving at Eq. (13) for $\langle[\Delta x(t)]^2\rangle$. Considering the equation for the probability distribution Eq. (3), we write down the expression for $P(x_2,t_2;x_1,t_1)$, the probability of finding a value x_2 at $t=t_2$ if the value was x_1 at $t=t_1$. We note the exact result

$$\int [x(t_2)-x(t_1)]^2 P(x_2,t_2;x_1,t_1)dx_2dx_1 = \langle[\Delta x(t_2-t_1)]^2\rangle, \tag{16}$$

whence

$$\langle x^2(t_2)\rangle + \langle x^2(t_1)\rangle - 2\langle x(t_2)x(t_1)\rangle = \langle[\Delta x(t_2-t_1)]^2\rangle, \tag{17}$$

and we can now use Eq. (13) to calculate $a(t_1,t_2) = \langle x(t_2)x(t_1)\rangle$. To obtain a Gaussian stationary process, it is necessary to calculate the correlation of $\bar{X}=x(t)/\sqrt{x^2(t)}$ and this leads to a complicated-looking expression. To express the final answer in a particularly simple form, we use the regime $\frac{t}{L^2}\ll 1$ and then exponentiate to find

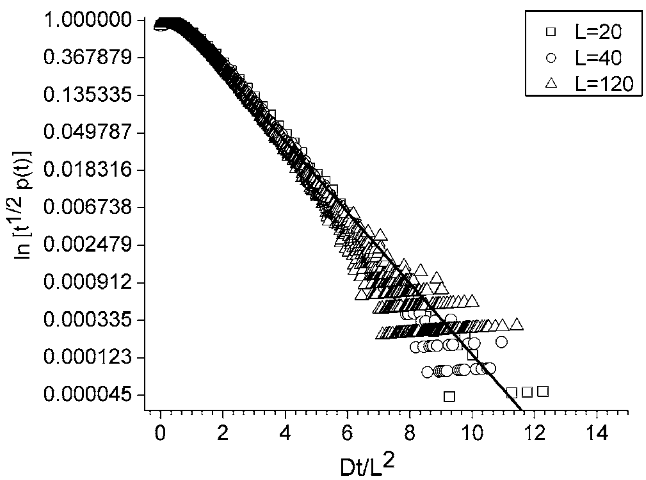

FIG. 1. Semilogarithmic plot of $t^{1/2}p(t)$ vs $\frac{Dt}{L^2}$. The straight line is the best fit for the linear part of the curve.

$$\langle \bar{X}(t_2)\bar{X}(t_1)\rangle = \sqrt{\frac{t_1}{t_2}}e^{-\gamma[D(t_2-t_1)/2L^2]}. \tag{18}$$

This exponentiation is predicted by the generic form anticipated by Redner. In the process of our calculation, we find the numerical prefactors that are not present in the general arguments of Redner. For comparison with the numerical simulations, these prefactors are essential. We now perform the transformation in time $t\to T=\ln(t)+\frac{\gamma Dt}{L^2}$. The correlator $f(T_2,T_1)$ in the transformed variables is

$$\langle \bar{X}(T_2)\bar{X}(T_1)\rangle = e^{-1/2(T_2-T_1)}. \tag{19}$$

The process is now a GSP. The survival probability is now given by

$$p(t,L) = t^{-1/2}e^{-\gamma Dt/2L^2} = t^{-1/2}f\left(\frac{t}{L^2}\right). \tag{20}$$

To test Eq. (20), we have calculated $p(t)$ numerically. Equation (20) can be recast as

$$t^{1/2}p(t,L) = e^{-\gamma Dt/2L^2}. \tag{21}$$

We expect the semilogarithmic plot of $t^{1/2}p(t,L)$ vs Dt/L^2 to be a straight line with a slope of $-\bar{\gamma}=-\gamma/2$. The value of $\bar{\gamma}$ obtained from our calculations is

$$\bar{\gamma} = \frac{\pi^2}{2}\left(1-\frac{\pi^2}{12}\right) = 0.8761. \tag{22}$$

A numerical simulation of the process was done using various values of L. The probability was obtained by averaging over 10^6 configurations. The numerically obtained value of $\bar{\gamma}$ is 0.9482. This discrepancy can be attributed to the approximate form of Eq. (18). A semilogarithmic plot of $t^{1/2}p(t,L)$ vs $\frac{Dt}{L^2}$ is shown in Fig. 1. This clearly shows the validity of Redner's generic form and the reasonableness of our approximations in arriving at the numerical value for $\bar{\gamma}$ and the fact that $z=2$.

We next consider a particle trapped in a harmonic potential and acted upon by a random force. Instead of sharp boundaries, we now have the potential confining the particle. If the confining length is L, then on dimensional grounds we expect $\omega\sim 1/L$. This is the setup of Ref. [14] except there the inertial effect cannot be ignored. The equation of motion is

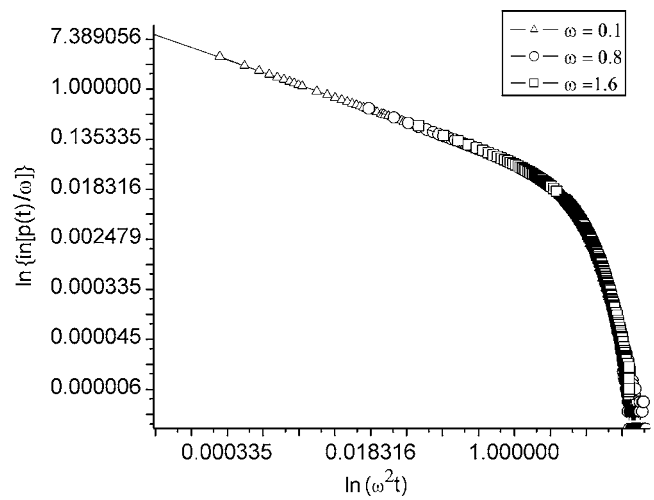

FIG. 2. Log-log plot of $p(t)$ vs $\omega^2 t$. The line shows the function $\frac{e^{-\omega^2 t/2}}{\sqrt{\sinh(\omega^2 t)}}$.

$$\frac{dx(t)}{dt} + \omega^2 x = \eta(t), \tag{23}$$

where $\eta(t)$ is the random noise whose moments are given by Eqs. (2a) and (2b) and ω is the strength of the harmonic potential. The expression for $x(t)$ then becomes

$$x(t) = e^{-\omega^2 t}\int_0^t e^{\omega^2 t'}\eta(t')dt'. \tag{24}$$

The correlator $\langle x(t_1)x(t_2)\rangle$ is given by

$$\langle x(t_1)x(t_2)\rangle = e^{-\omega^2(t_1+t_2)}\int_0^{t_1}\int_0^{t_2} e^{\omega^2(t_1'+t_2')}\langle \eta(t_1')\eta(t_2')\rangle dt_1'dt_2'. \tag{25}$$

Using Eq. (2b) we have

$$\langle x(t_1)x(t_2)\rangle = \frac{D}{2\omega^2}[e^{-\omega^2(t_1-t_2)} - e^{-\omega^2(t_1+t_2)}]. \tag{26}$$

The correlator in the new scaled variable $x(t)\to\bar{X}(t) = \frac{x(t)}{\sqrt{\langle x^2(t)\rangle}}$ has the form

$$\langle \bar{X}(t_1)\bar{X}(t_2)\rangle = e^{-\omega^2/2(t_1-t_2)}\left[\frac{\sinh(\omega^2 t_2)}{\sinh(\omega^2 t_1)}\right]^{1/2}. \tag{27}$$

Writing $e^T=1/\omega^2 e^{\omega^2 t}\sinh(\omega^2 t)$ we have

$$f(T_1,T_2) = \langle \bar{X}(T_1)\bar{X}(T_2)\rangle = e^{-\lambda(T_1-T_2)}, \tag{28}$$

where $\lambda=\frac{1}{2}$. The process is now a Gaussian stationary process. The survival probability can now be written as

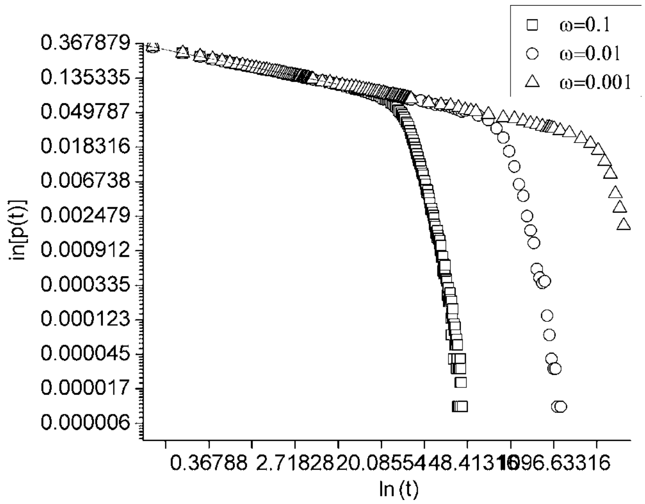

FIG. 3. Log-log plot of survival probability vs time. The solid line is the best-fit line. The estimated persistence exponent by best fit is $\theta=0.2502$.

$$p(T)=e^{-\lambda T}, \tag{29}$$

and in real time the survival probability is

$$p(t,\omega)=\left[\frac{1}{\omega^2}e^{\omega^2 t}\sinh(\omega^2 t)\right]^{-1/2}=\frac{\omega e^{-\omega^2 t/2}}{\sqrt{\sinh(\omega^2 t)}}=\frac{1}{t^{1/2}}f(\omega^2 t), \tag{30}$$

where

$$f(x)=\sqrt{\frac{x}{\sinh(x)}e^{-x}}. \tag{31}$$

As stated before $z=2$ and $f(x)\to 1$ as $x\to 0$, while $f(x)\to 0$ as $x\to\infty$. For $\omega\to 0$ the above expression for probability reduces to the normal random walk problem and the probability $p(t)$ goes as $t^{-1/2}$. The numerical data is obtained for three values of ω. For $t<1/\omega^2$, the estimated value of the exponent by fitting the log-log plot with a straight line is found to be $\theta=0.5055$. The numerically calculated values of $p(t,\omega)$ for various values of ω has been plotted in Fig. 2.

Finally, we present numerical studies of the survival probability for a harmonically trapped randomly accelerated particle and for a randomly accelerated particle with viscous drag. The persistence exponent for the randomly accelerated particle is $\theta=0.25$ [21–23]. The equation of motion for the particle is

$$\frac{d^2x}{dt^2}+\omega^2 x=\eta(t), \tag{32}$$

where $\eta(t)$ is a Gaussian white noise with a correlator given by Eqs. (2a) and (2b). Rescaling $\tau=\omega t$ we see that for $\tau\ll 1$ or $t\ll 1/\omega$ the first term dominates and the equation of motion is that of a randomly accelerated particle. A plot of survival probability vs time is shown in Fig. 3. With ω behaving as $\frac{1}{L}$ as noted above, this corresponds to $z=1$ for the dynamic component. The survival probability is obtained by averaging over 10^5 configurations.

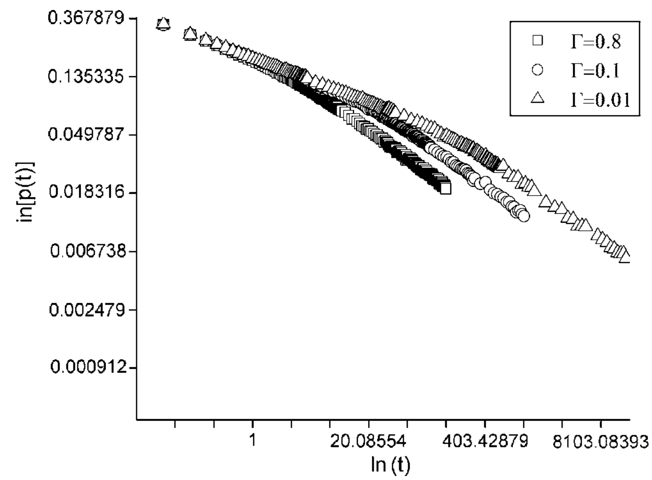

FIG. 4. Log-log plot of survival probability vs time.

For the randomly accelerated particle with viscous drag the equation of motion is

$$\frac{d^2x}{dt^2}+\Gamma\frac{dx}{dt}=\eta(t). \tag{33}$$

This is the crossover, which is of particular interest in the experiment of Ref [14]. As can be seen from our results the crossover in the form of the correlation function occurs at $t\sim\frac{1}{\Gamma}$. In this case, however, two regimes exists. For $t\ll 1/\Gamma$ the equation of motion is that of a randomly accelerated particle with the first term dominating and for $t\ll 1/\Gamma$ the second term dominates and the equation of motion is that of a random walker. Thus, the survival probability also shows a crossover from the randomly accelerated regime to random walk regime. The estimated values of the exponents in the two regimes is tabulated below.

Value of Γ	θ for $t<1/\Gamma$	θ for $t>1/\Gamma$
0.8	0.2638	0.4970
0.1	0.2510	0.4888
0.01	0.2538	0.4797

Numerically obtained values of survival probability are plotted against time in Fig. 4. The survival probabilities are obtained by averaging over 10^5 configurations. One of the important findings of Ref. [14] was that in the realistic situation of the experiment the crossover in $\langle(\Delta x)^2\rangle$ occurred for $t\gg\frac{1}{\Gamma}$. This was attributed to a memory-dependent damping term. In a future work, we will explore the effect of it on the persistence problem.

To conclude, we have studied both finite-size behavior and crossover behavior in some simple random walk situations where analytic expressions can be obtained to support the numerics. In particular, the finite-size study shows a power-law decay of the persistence for a large system and an exponential decay for small system size in conformity with the anticipated form of Redner [16], the experimental work on step size fluctuation [13] and the numerical work of Ref [17]. The crossover has been studied for an accelerated random motion and the crossover time is in agreement with the time where the ballistic motion crosses over to a random walk. It remains to be seen whether in the realistic situation of the experiment of Ref. [14], the crossover follows a similar time scale.

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